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Electrorotation of a leaky dielectric spheroid immersed in a viscous fluid

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We study rotation of a spheroidal dielectric particle immersed into a viscous dielectric fluid and subjected to a constant external electric field. The electrorotation is caused by the mechanical torque due to a fluid shear flow and by the electric moment due to the difference between electric conductivities and permeabilities of a particle and a host fluid. We consider effect of a shear flow on the orientation of the spheroid in a case of an ideal dielectric and for the finite electric conductivities of the spheroid and a host medium. We determined the critical magnitude of the shear flow velocity whereby spheroid cannot be held by the external electric field with a given strength and found the dependencies of the Euler angles of the spheroid vs the strength of the electric field. In a case of a system with a finite electric conductivity we considered two types of the nonideal dielectrics, namely, negative electroviscosity particles and positive electroviscosity particles. For a given magnitude of shear velocity and for two types of particles we determined the critical strengths of the electric field whereby rotation regime changes qualitatively in a case of rotation around the symmetry axis and in a case when the orientation of the axis of symmetry changes.

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I. INTRODUCTION

Dynamics of the particles in the external electric field was a subject of numerous investigations (see, e.g., monographs [1,2] and references therein). This topic of research attracted attention because of the continuous progress in the experimental techniques and necessity to adapt the existing and new models for explanation of the experimental results. One of the most widely used models in this field of research is a leaky dielectric model [3]. The latter model allows describing rotation of a particle subjected to a constant electric field around the axis of symmetry (Quincke rotation). The principal feature of the leaky dielectric model is that it accounts for the finite electric conductivity of a system. Depending upon the relation between the conductivity and permittivity of the host fluid, σ_1, ε_1, and electric conductivity and permittivity of a particle, σ_2, ε_2, the behavior of the particle strongly varies. In particular, when the following condition is satisfied:

$$\sigma_2/\sigma_1 > \varepsilon_2/\varepsilon_1, \tag{1}$$

the velocity of rotation of particle around the axis of symmetry is smaller than the local rotation velocity of the host fluid in the whole range of the parameters of the system. In the case when the inverse inequality is satisfied, depending on the external parameters of the system, i.e., fluid flow vorticity and amplitude of the electric field, the particle can rotate in different regimes whereby the rotation velocity of the particle is either larger or smaller than rotational velocity of the fluid depending upon the rotation regime. It was suggested in [4] to classify particles satisfying Eq. (1) as positive electroviscosity (PEV) particles, while particles satisfying the inverse inequality are called negative electroviscosity (NEV) particles. This terminology is motivated by that in the rotating external electric field the PEV particles rotate in the direction of rotation of the field while the NEV particles, in the main regime of their rotation, rotate against the direction of rotation of the external field. The comprehensive description of this situation can be found in [4].

Another difference between the NEV and PEV particles is manifested during their relaxation to the equilibrium orientation. This problem in the case for NEV particles and without shear flow was considered in [5]. In the latter study it was showed that at certain magnitude of the amplitude of the electric field the equilibrium orientation of particle loses stability through Hopf bifurcation [6]. In the present study we investigate the effect of shear flow on the stability of the equilibrium orientation of a spheroidal particle in the case of PEV and NEV particles. It should be noted that relaxation of a spheroidal particle in strongly overdamped regime but in nonstationary electric field was analyzed in [[7–11].

In the present study we consider only the case of a homogeneous constant electric field. Under these conditions we investigate stability of orientation of a spheroid in the shear fluid flow and stability of Quincke rotation. In the case of the Quincke rotation there are two types of instabilities. The first one is the inertial instability, which is caused by the inertia moment of the particle. The second instability is the instability of the orientation of a rotating particle, which is associated with dependence of the dipole moment of a spheroid and its rotation frequency on the direction of particle axis with respect to the external electric field. These two types of instabilities without shear flow were investigated in [4,12,13] and in [14], correspondingly.

This paper is organized as follows. In Sec. II we describe a mathematical model for describing rotation of a spheroidal particle subjected to a mechanical torque and a torque caused by the external electric field. In this section we also present equations of relaxation of the dipole and demonstrate characteristic trajectories of particle rotation for different initial conditions for numerical illustration of the model. In Sec. III

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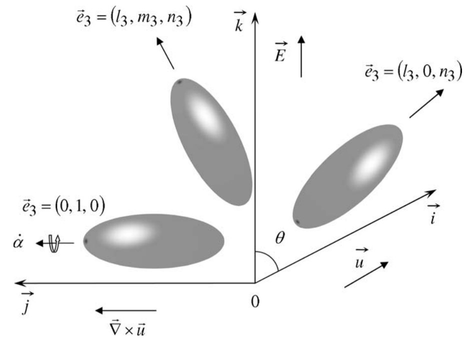

FIG. 1. Orientation of axis of symmetry in stationary and nonstationary regimes.

planar rotation of a spheroidal particle is analyzed in the case when the particle axis of symmetry lies in the plane spanned by the electric field strength vector and vector of fluid shear flow. In this section we determine the dependencies of the Euler angles of the spheroid vs strength of the electric field in a case of ideal dielectric and for a system with a finite electric conductivity. It is demonstrated that parameters of spheroid determine the magnitude of the maximum angle of its stable orientation. In a case of PEV particles and in an ideal dielectric case there always exists a sufficiently high magnitude of the electric field strength which allows us to keep the axis of spheroid in the direction with a stable orientation. Here increasing the strength of electric field allows to decrease the angle of inclination toward the value of inclination angle without shear flow. In a case of NEV particles a stable orientation exists only in a certain range of the strength of the electric field whereby the lower and the upper bounds depend upon the magnitude of shear. For a certain critical value of shear velocity these upper and lower bounds coincide, and at larger shear velocities the NEV particles cannot be in a state of rest.

In Sec. IV we investigate instability of orientation of various modes of Quincke rotation in the presence of the fluid shear flow. In Sec. V inertial stability of different modes of electrorotation around the axis of symmetry (Quincke rotation) is analyzed.

II. MATHEMATICAL MODEL

Consider an axially symmetric particle with permittivity ε_2 and conductivity σ_2 immersed in a viscous fluid with a coefficient of dynamic viscosity η, permittivity and conductivity ε_1 and σ_1, respectively. Assume that before the particle has been immersed in a host fluid, the velocity field in the fluid $\vec{u}$ reads

$$\vec{u}=\vec{i}\nu_0 z. \tag{2}$$

Hereafter $\vec{i}$ and $\vec{j},\vec{k}$ denote Cartesians unit vectors in the laboratory frame of reference, $\vec{k}=\vec{i}\times\vec{j}$ and z axis is directed along the vector $\vec{k}$ (see Fig. 1). The system is subjected to the dc-electric field with the amplitude E_0 and the strength of the electric field is $\vec{E}=E_0\vec{k}$. Rotation of an ellipsoidal particle in a shear flow with small Reynolds number without electric field was comprehensively investigated in [15]. According to the results obtained in [15] (taking into account a different choice of coordinate system in the present study, see Appendix A) the torque acting at the particle in a frame of reference attached to the particle reads

$$\vec{M}=M_i\vec{e}_i,$$

where $\vec{e}_i$ is the unit vectors directed along the principal axes of the particle and

$$M_i=2\eta V M_{0i}\left(\frac{\vec{e}_i\cdot\vec{j}\nu_0}{2}-\omega_{0i}+\Gamma_i c_i\right), \tag{3}$$

ω_{0i} are components of the angular velocity of the ellipsoid along the principal axes, V is a volume of the particle, M_{0i}, Γ_i, c_i are coefficients depending on the velocity $\vec{u}$ and on the geometric parameters of the ellipsoid which are specified below. Relation between the coordinate systems adopted in this study and in [15] is presented in Appendix A. Neglecting particle inertia (as it is done in the main part of this study) equations determining angular velocity of the particle without electric field can be written as $M_i=0$. In the presence of electric field the latter equation is replaced by $M_i+T_i=0$. Here $\vec{T}$ is an electrical torque caused by the external field, $\vec{T}=\vec{P}\times\vec{E}$ (for details see [3,16]) and $\vec{P}$ is a dipole moment of the particle which is specified below. Consider a spheroidal particle with the axis of symmetry directed along a unit vector $\vec{e}_3$ and with the lengths of semiaxes a_1,a_2,a_3, where a_3 is the length of semiaxis along the axis of symmetry and $a_1=a_2=a_\perp$ are the lengths of the perpendicular semiaxes. Hereafter, we use the frame attached with a rotating spheroidal particle with unit coordinate vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$ such that $\vec{e}_1\times\vec{e}_2=\vec{e}_3$. Using notation $\beta=a_3/a_\perp$ the coefficients Γ_i and c_i in Eq. (2) can be written as follows (see [15] and Appendix A):

$$\Gamma_1=-\Gamma_2=\frac{1-\beta^2}{1+\beta^2},\quad \Gamma_3=0,\quad c_i=\frac{\nu_0}{2}e_{ikl}^2 l_k n_l, \tag{4}$$

where l_k are components of the unit vector $\vec{i}$ along the principle axes of the spheroid, $l_k=\vec{i}\cdot e_k$, n_l are components of the unit vector $\vec{k}$ along the principle axes of the spheroid, $n_l=\vec{k}\cdot e_l$, e_{ikl} is Levi-Civita symbol, $e_{ikl}=e_i(e_k\times e_l)$, $e_{ikl}^2=[e_i(e_k\times e_l)]^2$. Formulas for coefficients M_{0i} can be written using as a parameter a depolarization factor along the axis of symmetry, d. In case of a prolate spheroid $\beta>1$ and $d=(1-e^2)[th^{-1}(e)-e]/e^3$, $e=\sqrt{1-1/\beta^2}$. In the case of an oblate spheroid $\beta<1$ and $d=(1+e^2)[e-tg^{-1}(e)]/e^3$, where $e=\sqrt{1/\beta^2-1}$. Using a depolarization factor as a parameter, formula for M_{0i} can be written as follows:

$$M_{01}=\frac{(1+\beta^2)}{\frac{1-d}{2}+d\beta^2},\quad M_{02}=M_{01},\quad M_{03}=\frac{2}{1-d}. \tag{5}$$

Equation (5) elucidates the dependence of parameters M_{0i} on parameter β taking into account that for a prolate spheroid parameter d varies in the range $0<d<1/3$ and for an oblate spheroid $1/3<d<1$.

Dipole moment of a particle with a finite electric conductivity $\vec{P}=\vec{P}_\varepsilon+\vec{P}_\sigma$, where $\vec{P}_\varepsilon$ is a dipole moment caused by local polarization which coincides with the dipole moment in the case of an ideal dielectric [16,17]:

$$\vec{P}_\varepsilon = \varepsilon_0 V \sum_{i=1}^{3} \frac{E_0 n_i \vec{e}_i \kappa_\varepsilon}{(1+f_{i\varepsilon})}. \tag{6}$$

Here $\kappa_\varepsilon=\varepsilon_2/\varepsilon_1-1$, $f_{i\varepsilon}=\kappa_\varepsilon d_i$, ε_0 is permittivity of vacuum, and depolarization coefficients d_i in the case of a spheroidal particle are determined by the following formula:

$$d_1 = d_2 = \frac{1-d}{2}, \quad d_3 = d.$$

Dipole moment $\vec{P}_\sigma$ is caused by a finite conductivity and is associated with accumulation of electric charge at the particle-host fluid interface:

$$\vec{P}_\sigma = \varepsilon_0 V \sum_{i=1}^{3} \frac{E_0 \Pi_i(t) \vec{e}_i (\kappa_\sigma - \kappa_\varepsilon)}{(1+f_{i\varepsilon})(1+f_{i\sigma})}, \tag{7}$$

where $\kappa_\sigma=\sigma_2/\sigma_1-1$, $f_{i\sigma}=\kappa_\sigma d_i$, and functions $\Pi_i(t)$ are determined by the following equation:

$$\dot{\Pi}_i + \frac{\Pi_i}{\tau_i} = \frac{n_i}{\tau_i}, \quad \tau_i = \frac{1+f_{i\varepsilon}}{1+f_{i\sigma}}. \tag{8}$$

Hereafter, notation $\dot{A}$ denotes a derivative of parameter A with respect to dimensionless time t/τ_0, where $\tau_0=\varepsilon_0\varepsilon_1/\sigma_1$ is a characteristic charge relaxation time. Derivation of expression for the dipole moment is presented in Appendix B. In Appendix B we also derive equations which yield formulas (7) and (8) and discuss different definitions of the dipole moment adopted in the literature (see also [10,11]). In particular, the definition of the dipole moment used in this study implies the absence of coefficient ε_1 in Eqs. (6) and (7). As for the rest, the system of electrodynamic equations used in this study is essentially identical (the difference is only in notations) to the system of equations which was derived first in [18].

Relation $M_i+T_i=0$ yields equations determining components of particle rotation velocity along the principal axes of the spheroid:

$$\omega_{0i} = \frac{m_i \nu_0}{2} + \Gamma_i c_i + \frac{F_i R_0}{2M_{0i}\tau_0} e_{ikl}^2 n_k n_l - \frac{R_0}{M_{0i}\tau_0} e_{ikl}\chi_k \Pi_k n_l, \tag{9}$$

where

$$F_1 = -F_2 = \frac{(3d-1)\kappa_\varepsilon^2}{2(1+f_{1\varepsilon})(1+f_{3\varepsilon})}, \quad F_3 = 0, \quad R_0 = \frac{E_0^2}{E_*^2},$$

$$E_*^2 = \frac{2\eta}{\varepsilon_0 \tau_0}, \quad \chi_i = \frac{\kappa_\varepsilon - \kappa_\sigma}{(1+f_{i\varepsilon})(1+f_{i\sigma})}.$$

Coefficients l_i, m_i, n_i form an orthogonal matrix $\hat{O}$ which relates unit coordinate vectors in the coordinate system attached with the particle and those in the laboratory frame. The elements of this matrix can be written as follows:

$$O_{i1} = l_i, \quad O_{i2} = m_i, \quad O_{i3} = n_i. \tag{10}$$

Components of particle rotation velocity can be expressed using this matrix:

$$2\omega_{0i}\tau_0 = 2\omega_i = e_{ikl}\Omega_{kl}, \tag{11}$$

where an antisymmetric tensor $\hat{\Omega}$ is determined by the following formula:

$$\hat{\Omega} = \dot{\hat{O}} \cdot \hat{O}^T \text{ and } \hat{O}\cdot\hat{O}^T = 1. \tag{12}$$

Using Euler angles α, θ, ϕ matrix $\hat{O}$ can be written as $\hat{O} = \hat{O}_3(\alpha)\hat{O}_2(\theta)\hat{O}_1(\phi)$ (see [19]), where formulas for $\hat{O}_1, \hat{O}_2, \hat{O}_3$ are presented in Appendix A. Comparing expression (10) for matrix $\hat{O}$ and the expression for this matrix through Euler angles α, θ, ϕ allows determining coefficients l_i, m_i, n_i (see Appendix A) as functions of the angles α, θ, ϕ. Using these expressions and Eqs. (11) and (12) we find that

$$\omega_1 = \dot{\phi} \sin\theta \sin\alpha + \dot{\theta}\cos\alpha,$$

$$\omega_2 = \dot{\phi}\sin\theta\cos\alpha - \dot{\theta}\sin\alpha, \quad \omega_3 = \dot{\phi}\cos\theta + \dot{\alpha}. \tag{13}$$

Equations (9) and (13) yield a system of differential equations:

$$\dot{\theta} = \frac{\bar{\nu}}{2}\sin\phi(1-\Gamma_1\cos 2\theta) + \frac{R_0}{M_{01}} \times \left(F_1 \frac{\sin 2\theta}{2} - \chi_1 \Pi_\theta \cos\theta + \chi_3 \Pi_3 \sin\theta\right), \tag{14}$$

$$\dot{\phi} = \frac{\bar{\nu}}{2}(1-\Gamma_1) ctg\theta \cos\phi + \frac{R_0}{M_{01}}\chi_1 \Pi_N ctg\theta, \tag{15}$$

$$\dot{\alpha} = -\frac{\bar{\nu}\cos\phi}{2\sin\theta}(1-\Gamma_1\cos^2\theta) - \frac{\chi_1 R_0 \Pi_N}{\sin\theta}\left(\frac{\sin^2\theta}{M_{03}} + \frac{\cos^2\theta}{M_{01}}\right), \tag{16}$$

where

$$\bar{\nu} = \nu_0\tau_0, \quad \Pi_N = \Pi_1\cos\alpha - \Pi_2\sin\alpha,$$

$$\Pi_\theta = \Pi_1 \sin\alpha + \Pi_2\cos\alpha. \tag{17}$$

Set of six equations (8) and Eqs. (14)–(16) is a closed system of equations. Trajectory of the orientation of spheroid is determined by l_3, m_3, n_3. For numerical illustration of the derived equations let us consider a case with a zero initial charge of the particle. In this case the initial dipole moment of spheroid is determined by Eq. (6), and the time of formation of this dipole moment is related with a characteristic time of polarization. This time is negligibly small in comparison with characteristic time of the problem, τ_0, and the initial value of the dipole moment (6) for a given direction of the external electric field is uniquely determined by the initial value of $\vec{e}_3(0)=[l_3(0), m_3(0), n_3(0)]$. The assumption about the zero initial free charge at the surface of the spheroid corresponds to the choice $\Pi_i(0)=0$. In Fig. 2 we showed

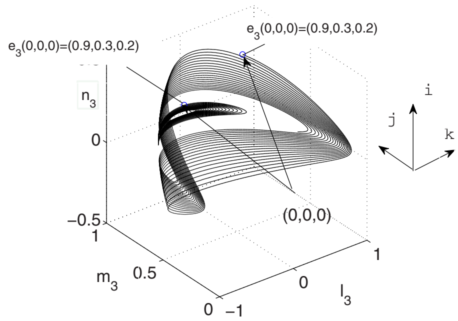

FIG. 2. (Color online) Variation in orientation of the axis of symmetry of spheroid for different initial conditions. Parameters of spheroid: $\kappa_\varepsilon=3$, $\kappa_\sigma=-0.7$, $\beta=15$. External parameters: $R_0=2$, $\bar{\nu}=10$.

trajectories of the orientation of a spheroid for different values of the initial orientation of the axis of symmetry of the spheroid with respect to the direction of the external electric field, $\vec{k}$, and the direction of the velocity of a shear flow, $\vec{i}$.

Now let us determine and investigate the stationary states of the system. In order to simplify notations we will use dynamic parameters Π_N,Π_θ determined by Eq. (17). Equation (8) for these parameters can be rewritten as follows:

$$\dot{\Pi}_N+\dot{\alpha}(\phi,\theta,t)\cdot\Pi_\theta=-\frac{\Pi_N}{\tau_1},$$

$$\dot{\Pi}_\theta-\dot{\alpha}(\phi,\theta,t)\cdot\Pi_N=-\frac{\Pi_\theta}{\tau_1}+\frac{\sin\theta}{\tau_1}, \tag{18}$$

$$\dot{\Pi}_3+\frac{\Pi_3}{\tau_3}=\frac{\cos\theta}{\tau_3}. \tag{19}$$

System of equations (18) and (19) is equivalent to system of equations (8). In the absence of the shear flow, $\bar{\nu}=0$, system of equations (14)–(16) and Eqs. (18) and (19) are identical to system of equations (11a)–(11c) and (12a)–(12c) in [14] (the difference is only in notations). The author of the latter study considered instability of orientation of the spheroid which rotates with a constant angular velocity around the axis of symmetry. In the present study this problem is considered in Sec. IV. In the next section we determine a condition for the existence of an equilibrium orientation of a spheroid and its stability under the action of a constant electric field. The case without shear flow was analyzed in [5].

III. PLANAR ROTATION OF A SPHEROID AND ITS STABILITY

Consider system of equations (14)–(16) and Eqs. (18) and (19). For determining the equilibrium orientation we set the rate of change of dynamic variables equal to zero. Condition $\dot{\alpha}=0$ and Eq. (17) imply that $\Pi_N=0$, $\Pi_\theta=\sin\theta$, $\Pi_3=\cos\theta$. Then Eqs. (15) and (16) yield $\phi=\pi/2+k\pi$ and Eq. (14) can be written as follows:

$$\frac{\bar{\nu}}{2}(1-\Gamma_1\cos 2\theta)+\frac{R_0\kappa_\sigma^2}{M_{01}2(1+f_{1\sigma})(1+f_{3\sigma})}\frac{3d-1}{2}\sin 2\theta=0. \tag{20}$$

In deriving Eq. (20) we used the following formula:

$$F_1+\chi_3-\chi_1=\kappa_\sigma^2\frac{3d-1}{2(1+f_{1\sigma})(1+f_{3\sigma})}. \tag{21}$$

The torque caused by the electric field is given by the second term in Eq. (20) and it does not depend on the parameter $\kappa_\varepsilon=\varepsilon_2/\varepsilon_1-1$. The reason for this behavior is that the torque acting at the particle for $t\to\infty$ in a system with a finite electric conductivity is determined by a finite surface charge and not by the initial polarization. This facet can be elucidated using the formulas for the total torque in [9,10]. Note that formula for the torque given by the second term in Eq. (20) can be obtained from the expression for the strength of the electric field $\vec{E}$ determined from the equations $\vec{\nabla}\cdot\vec{j}=0$, $\vec{j}=\sigma\vec{E}$, $E=-\vec{\nabla}\varphi$. Formulas for l_i,m_i,n_i show that the condition $\phi=\pi/2+k\pi$ yields $(l_3,m_3,n_3)=(\pm\sin\theta,0,\cos\theta)$. This corresponds to the case when the axis of the spheroid is in the plane spanned by velocity vector $\vec{i}$ and electric field vector $\vec{k}$ (see Fig. 1). When $\theta=0$ the axis of the spheroid is normal to the shear flow velocity while for $\theta=\pi/2$ the axis of the spheroid is directed along the shear flow velocity. The condition of existence of a static orientation of the spheroid corresponds to the condition for existence of the real roots of Eq. (20):

$$\Gamma_1^2+\frac{S^2}{\bar{\nu}^2}>1\quad\text{or}\quad S^2>\frac{\bar{\nu}^2 4\beta^2}{(1+\beta^2)^2}, \tag{22}$$

where

$$S=\frac{R_0\kappa_\sigma^2(1-3d)}{2M_{01}(1+f_{1\sigma})(1+f_{3\sigma})}.$$

The solution of Eq. (20) can be written as follows:

$$tg\,\theta=\frac{1+\beta^2}{2}\left[\frac{S}{\bar{\nu}}\mp\sqrt{\left(\frac{S}{\bar{\nu}}\right)^2-\frac{4\beta^2}{(1+\beta^2)^2}}\right]. \tag{23}$$

Formula (23) shows that there exist two equilibrium roots θ_1,θ_2 such that when $\bar{\nu}\to 0$, $\theta_1\to 0$, $\theta_2\to\pi/2$. If $S/\bar{\nu}>0$, the value θ_1 corresponds to the minus sign in Eq. (23). In this study we show that one of the solutions determined by Eq. (23) is always unstable.

Consider a case of a prolate spheroid and $\bar{\nu}>0$. In this case solution θ_2 is unstable. The equilibrium orientation of the spheroid is determined by the parameters $R_0/\bar{\nu}$, κ_σ, β. The expression $S/\bar{\nu}=2\beta/(1+\beta^2)$ determines a critical value, $(R_0/\bar{\nu})_{\min}=K$:

$$K(\beta,\kappa_\sigma)=\frac{2M_{01}(1+f_{1\sigma})(1+f_{3\sigma})2\beta}{(1-3d)(1+\beta^2)\kappa_\sigma^2}, \tag{24}$$

which separates on the plane $(\beta,R_0/\bar{\nu})$ the domain where equilibrium state exists, $R_0/\bar{\nu}>K$, from the domain without the equilibrium state, $R_0/\bar{\nu}<K$. The points of intersection of a straight line $(R_0/\bar{\nu})=\text{constant}$ with a curve $(R_0/\bar{\nu})_{\min}=K$

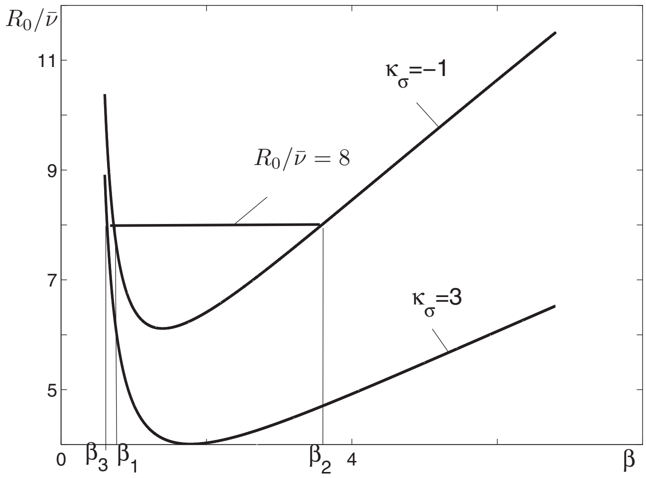

FIG. 3. Minimum value of the ratio $K=(R_0/\bar{\nu})_{\min}$ in the domain of static orientation of spheroid vs parameter β. For $R_0/\bar{\nu}<8$ static orientation exists only in the interval $\beta_1<\beta<\beta_2$, where β_1,β_2 are limiting values of parameter β for $(R_0/\bar{\nu})=8$, $\kappa_\sigma=-1$. The left boundary of the parameter β for $(R_0/\bar{\nu})=8$, $\kappa_\sigma=3$ is denoted by β_3.

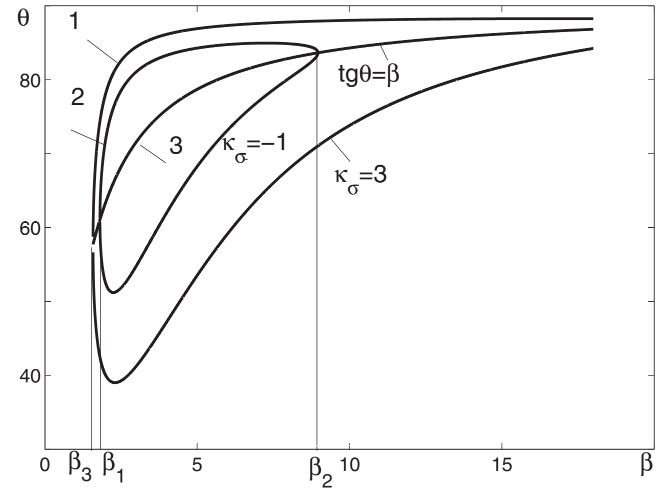

FIG. 4. Dependence of the angle of static orientation vs parameter β for $(R_0/\bar{\nu})=8$. Curve 1—$\kappa_\sigma=3$; curve 2—$\kappa_\sigma=-1$; curve 3—$tg\,\theta=\beta$.

are the points where different branches of Eq. (23) merge. In this case

$$tg\,\theta_1(\beta)=tg\,\theta_2(\beta)=\beta. \tag{25}$$

Since $\beta>1$ for a prolate spheroid, the merging angle is always large than $\pi/4$ for a prolate spheroid. Consequently, the limiting value of the angle whereby spheroid preserves its stable orientation is determined by the ratio of its semiaxes. In this case the limiting value satisfies the inequality, $\theta_{1\,\max}>\pi/4$. Because of the nonmonotonic character of the function $(R_0/\nu)_{\min}$ vs parameter β (see Fig. 3) there exist two merging angles for a given magnitude of R_0/ν. This implies that the same value of the limiting angle of orientation of the spheroid is attained for different values of the parameter β. In Fig. 4 we showed the dependence of the equilibrium orientations of the spheroid vs parameter β. Solution (23) with plus sign corresponds to the curves 1 and 2. The curve $\theta=tg^{-1}(\beta)$ intersects with these curves in their merge points according to Eq. (25).

It must be emphasized that if initially the particles are not charged then the formula for the torque T which orients the particles in the direction of the electric at the initial moment, $T(0)$, reads

$$T(0)=\frac{\varepsilon_0\kappa_\varepsilon^2E_0^2V}{2(1+f_{1\varepsilon})(1+f_{3\varepsilon})}\frac{3d-1}{2}\sin 2\theta=0.$$

The latter expression replaces the second term in Eq. (20) when determining orientation of the spheroid in a shear flow in the case of ideal dielectric. All the results obtained above are valid for the case of an ideal dielectric after substituting in all formulas $\kappa_\sigma\to\kappa_\varepsilon$.

Now let us analyze the stability of the determined solutions. Hereafter, we assume that a particle can move only in the plane spanned by vectors $\vec{E},\vec{u}$; i.e., the condition $\vec{e}_3\cdot\vec{j}=0$ is satisfied. For investigating stability of the system of Eqs. (14)–(16), (18), and (19) in the vicinity of the equilibrium state let us linearize these equations with respect to the parameters $\theta_1=\theta-\theta_S$, $\Pi_\theta^1=\Pi_\theta-\sin\theta_S$, $\Pi_3^1=\Pi_3-\cos\theta_S$. Then Eqs. (14)–(16), (18), and (19) yield

$$\dot{\theta}_1=a_1\theta_1+a_2\Pi_\theta^1+a_3\Pi_3^1,\quad \tau_1\dot{\Pi}_\theta^1+\Pi_\theta^1=\cos\theta_S\cdot\theta_1,$$

$$\tau_3\dot{\Pi}_3^1+\Pi_3^1=-\sin\theta_S\theta_1, \tag{26}$$

where

$$a_1=\bar{\nu}\Gamma_1\sin 2\theta_S+\frac{R_0}{M_{01}}(F_1\cos 2\theta_S+\chi_1\sin^2\theta_S+\chi_3\cos^2\theta_S),$$

$$a_2=-\frac{R_0}{M_{01}}\chi_1\cos\theta_S,\quad a_3=\frac{R_0}{M_{01}}\chi_3\sin\theta_S.$$

In [5] it was showed that the orientation of a prolate spheroid in the direction of the electric field in systems with finite electric conductivity can be unstable even without shear flow ($\bar{\nu}=0$). This occurs for sufficiently large amplitudes where the inverse to Eq. (1) inequality is satisfied, i.e., when spheroid is a NEV particle. In a case of PEV particles this instability does not occur. Without shear flow the PEV particle as in the case of an ideal dielectric has stable orientation which for a prolate spheroid coincides with the direction of the electric field. Clearly, the presence of a shear flow changes the situation because for sufficiently high flow velocity the particle cannot be held by the electric field. Since the loss of stability caused by high shear flow velocity is not strongly affected by relaxation times of dipole moments, τ_1,τ_3, these characteristic times can be considered small so that condition (27) is satisfied:

$$\tau_1\dot{\theta}\ll 1,\quad \tau_3\dot{\theta}\ll 1. \tag{27}$$

In the zero-order approximation in these parameters we arrive at the following equation:

$$\dot{\theta}_1 = \bar{\nu}\left(\Gamma_1 \sin 2\theta_S - \frac{S}{\nu}\cos 2\theta_S\right)\theta_1,$$

and condition for stability $\Gamma_1 \sin 2\theta_S < \frac{S}{\bar{\nu}}\cos 2\theta_S$. In the case of a prolate spheroid $\Gamma_1 < 0, S/\bar{\nu} > 0$. The equilibrium state is stable for $\theta_S < \pi/4$. When $\theta_S > \pi/4$, the stability conditions yield $2tg\,\theta_S/(tg^2\theta_S - 1) > S/\bar{\nu}|\Gamma_1|$. The latter relation and Eq. (23) imply that the equilibrium state θ_1 is always stable while the equilibrium state θ_2 which corresponds to a plus sign before the square root in Eq. (23) is always unstable. This observation was already mentioned above. Consequently, the domain of stability of a static orientation of a spheroid in the case of an ideal dielectric and PEV particle coincides with the domain of existence of real roots of Eq. (23). Condition (22) determines the amplitude of the external electric field,

$$R_{0b}(\bar{\nu}) = K(\kappa_\varepsilon, \kappa_\sigma, \beta)\bar{\nu}, \tag{28}$$

which is required for preventing PEV particle or an ideal dielectric spinning by a fluid flow.

In a case of NEV particles there exists an additional domain of instability. Physical nature of this instability is the same as for Quincke rotation. Indeed, at sufficiently high amplitudes of the electric field the dipole moment of the spheroid is directed against the direction of the external field due to the change of a surface charge, and static orientation of the spheroid becomes unstable. This problem without shear flow was studied in [5]. The presence of shear flow introduces new peculiarities. While in the case without shear flow the NEV-particle stability loss occurs at large strength of the applied external electric field, in the presence of shear flow with a relatively small velocity the NEV particle does not have static orientation for any amplitude of the electric field. Since stability loss by NEV-particle occurs at small shear flow velocities it can be assumed that $\bar{\nu} \ll R_0$, i.e., $S \gg \bar{\nu}2\beta/(1+\beta^2)$. As before let us consider a case of a prolate spheroid whereby for strong electric fields two roots satisfy the following equations: $tg\,\theta_1 = \beta^2\bar{\nu}/S(1+\beta^2) \ll 1$ and $tg\,\theta_2 = S(1+\beta^2)/\bar{\nu}$. Due to the smallness of the angle $\theta_S = \theta_1$ the characteristic equation corresponding to system of equations (24) can be written as follows:

$$D(\gamma) = D_0(\gamma) - \frac{\bar{\nu}^2}{R_0}D_1(\gamma) = 0,$$

where γ is a growth rate increment of functions $[\theta_1(t), \Pi^1_\theta, \Pi^1_3] \propto e^{\gamma t}$. Expressions for $D_0(\gamma)$ and $D_1(\gamma)$ read

$$D_0(\gamma) = (\gamma + 1/\tau_3)\left[\gamma^2 + \left(\frac{1}{\tau_1} - a_1^0\right)\gamma + \frac{S}{\tau_1}\right],$$

$$D_1(\gamma) = b_1\gamma^2 + \left(\frac{b_1 - b_3}{\tau_3} + \frac{b_1 + b_2}{\tau_1}\right)\gamma + \frac{b_1 + b_2 - b_3}{\tau_3\tau_1},$$

where

$$a_1^0 = \frac{R_0}{M_{01}}(F_1 + \chi_3), \quad b_1 = 2\Gamma_1\bar{\theta}_1 - \frac{2\bar{\theta}_1^2}{M_{01}}\left(F_1 + \frac{\chi_3 - \chi_1}{2}\right),$$

$$b_2 = \frac{\chi_1\bar{\theta}_1^2}{M_{01}}, \quad b_3 = \frac{\chi_3\bar{\theta}_1^2}{M_{01}}. \tag{29}$$

Parameters $\bar{\theta}_1$ and b_i do not depend on R_0, $\bar{\nu}$ and $\bar{\theta}_1$ is determined by the following formulas:

$$tg\,\theta_1 \approx \theta_1 = \nu\bar{\theta}_1/R_0, \quad \bar{\theta}_1 = \frac{1}{2}K(\beta)\beta,$$

where $K(\beta)$ is determined by Eq. (24)

The first corrections to the roots of equation $D(\gamma)=0$ can be determined from the following equation:

$$D_0(\gamma_{0i} + \gamma_i') = \frac{\bar{\nu}^2}{R_0}D_1(\gamma_{0i}), \tag{30}$$

where γ_{0i} are three roots of the equation $D_0(\gamma)=0$, and γ_i' is the first correction which is obtained after linearization of Eq. (30) with respect to this parameter. Let us denote the roots of equation $D_0(\gamma)=0$ as follows:

$$\gamma_{01} = -\frac{1}{\tau_3}, \quad 2\gamma_{0\pm} = a_1^0 - \frac{1}{\tau_1} \pm i\sqrt{\frac{4S}{\tau_1} - \left(a_1^0 - \frac{1}{\tau_1}\right)^2}. \tag{31a}$$

The value of the amplitude

$$R_{0c}(0) = \frac{M_{01}}{F_1 + \chi_3}\frac{1}{\tau_1}, \tag{31b}$$

which is implied by the condition $a_1^0 = \frac{1}{\tau_1}$, separates between the domain of stability of the equilibrium state $R_0 < R_{0c}(0)$ and the instability domain $R_0 > R_{0c}(0)$ in a case without a shear flow $\bar{\nu}=0$. The condition of existence of the state $R_{0c}(0)$ reads

$$F_1 + \chi_3 = \frac{\kappa_\varepsilon - \kappa_\sigma + \kappa_\varepsilon\kappa_\sigma(3d-1)/2}{(1+f_{1\varepsilon})(1+f_{3\sigma})} > 0. \tag{32}$$

For $R_0 = R_{0c}(0)$ the roots of the equation $\gamma_{0\pm}$ are imaginary:

$$\gamma_{0\pm} = \pm i\sqrt{S^0/\tau_1}, \quad S^0 = \frac{\kappa_\sigma^2(1-3d)}{2(\kappa_\varepsilon - \kappa_\sigma) + \kappa_\sigma\kappa_\varepsilon(3d-1)}. \tag{33}$$

Equation (32) is valid only when the inverse inequality to Eq. (1) is satisfied, i.e., if $\kappa_\varepsilon > \kappa_\sigma$ (NEV particles). The first corrections $\gamma_\pm', \gamma_1'$ read

$$\gamma_1' = \frac{\bar{\nu}^2}{R_0}\frac{D_1\left(-\frac{1}{\tau_3}\right)}{\frac{1}{\tau_3^2} + \left(a_1^0 - \frac{1}{\tau_1}\right)\frac{1}{\tau_3} + \frac{S}{\tau_1}},$$

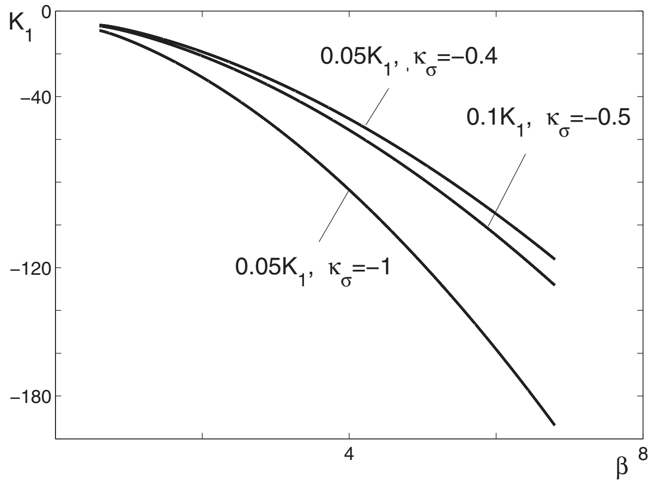

FIG. 5. Ratio K_1 in the domain of static orientation vs parameter β for $\kappa_\varepsilon=0$.

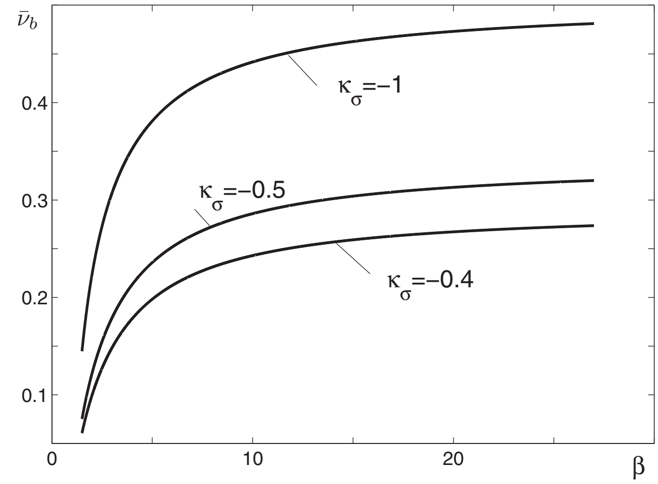

FIG. 6. Maximum value of parameter $\bar{\nu}=\bar{\nu}_b(\beta)$ in the domain of static orientation of spheroid vs parameter β for $\kappa_\varepsilon=0$. For $\bar{\nu} > \bar{\nu}_b(\beta)$ static orientation does not exist.

$$\gamma'_{\pm} = \mp i\frac{\bar{\nu}^2}{R_0}\frac{D_1(\gamma_{0\pm})}{\left(\gamma_{0\pm}+\frac{1}{\tau_3}\right)\sqrt{\frac{4S}{\tau_1}-\left(a_1^0-\frac{1}{\tau_1}\right)^2}}. \quad (34)$$

Condition for loss of stability corresponds to vanishing of the real parts of the roots of dispersion equation (30):

$$\mathrm{Re}(\gamma_{0+}+\gamma'_+) = \mathrm{Re}(\gamma_{0-}+\gamma'_-) = 0. \quad (35)$$

Consequently, the NEV particles are stable in the following range of the amplitudes of the external electric field:

$$R_{0b}(\bar{\nu}) < R_0 < R_{0c}(\bar{\nu}), \quad (36)$$

where $R_{0c}(\bar{\nu})$ is determined by the following formula:

$$R_{0c}(\bar{\nu}) = R_{0c}(0) + K_1(\kappa_\varepsilon,\kappa_\sigma,\beta)\bar{\nu}^2 \quad (37)$$

and K_1 is determined by Eq. (35)

Substituting expression (37) in Eq. (35) and keeping the terms with the order not higher than $\bar{\nu}^2$ yields

$$K_1(\kappa_\varepsilon,\kappa_\sigma\beta) = \frac{\tau_1^{3/2}}{\sqrt{4S^0}}\mathrm{Im}\left(\frac{D_1\{\gamma_{0+}[R_{0c}(0)]\}}{\gamma_{0+}[R_{0c}(0)]+1/\tau_3}\right). \quad (38)$$

Hence, Eqs. (36) and (37) together with expression (38) determine the domain of stability of NEV particles. In Fig. 5 we showed the dependence of parameter $K_1(\kappa_\sigma,\beta)$ vs β for different $\kappa_\sigma<0$ and $\kappa_\varepsilon=0$. Note that when $\kappa_\varepsilon=0$, the condition for a particle being a NEV particle is $\kappa_\sigma<0$. Inspection of Fig. 5 shows that the magnitude of $K_1(\kappa_\sigma,\beta)$ strongly depends on the parameter κ_σ. In order to present all curves on the same figure we used the scale factors as shown in Fig. 5. Strong dependence of K_1 vs β and its negative range of variation, $K_1<0$, indicate that for already small values of $\bar{\nu}$ the upper bound of inequality (36) merges with the lower bound of this inequality. The point of merging, $\bar{\nu}_b$, is determined by a condition $R_{0b}(\bar{\nu}_b)=R_{0c}(\bar{\nu}_b)$ or

$$\bar{\nu}_b = -\frac{K(\beta)}{2|K_1(\beta)|} + \sqrt{\frac{K^2(\beta)}{4K_1^2}+\frac{R_{0c}(0)}{|K_1|}}. \quad (39)$$

When the magnitude of the velocity $\bar{\nu}>\bar{\nu}_b(\beta)$, the NEV particle cannot remain in the equilibrium state in the plane spanned by vectors $\vec{E}$ and $\vec{u}$ at any amplitude of the external electric field R_0. In Fig. 6 we showed the dependence $\bar{\nu}_b(\kappa_\varepsilon,\kappa_\sigma,\beta)$ vs β. Inspection of this figure shows that for already small absolute values of the vorticity of a shear flow $\bar{\nu}$, the equilibrium orientation of the NEV particle becomes unstable. Since the loss of stability occurs for small values of the parameter $\bar{\nu}$, this justifies the used above procedure for determining the boundary of the stability domain based on expansion in powers of small parameter. Knowing the value of $R_{0c}(\bar{\nu})$ allows us to determine the "principal" oscillation frequency in the vicinity of the point of a stability loss, $R_0=R_{0c}(\bar{\nu})$ (domain of weak nonlinearity). This frequency ϖ is given by the following formula:

$$\varpi = \sqrt{\frac{S^0}{\tau_1}} - \frac{\bar{\nu}^2}{2R_{0c}(0)\sqrt{S^0/\tau_1}}\mathrm{Re}\left(\frac{D_1\{\gamma_{0+}[R_{0c}(0)]\}}{\gamma_{0+}[R_{0c}(0)]+1/\tau_3}\right).$$

Hence, we demonstrated that for given parameters of the problem, $\bar{\nu}$, κ_ε, κ_σ, β, there exist two critical points, $R_{0b}(\bar{\nu})$ and $R_{0c}(\bar{\nu})$, for the NEV particle. In the vicinity of a lower bound $R_0>R_{0b}$ the point of equilibrium on the phase plane $(\theta,\dot{\theta})$ is a node while in the vicinity of an upper bound $R_0<R_{0c}$ the equilibrium point is a focus. The amplitude of the external field, R_{0t}, whereby the character of the attractor changes is determined by the following relation:

$$1/\tau_1 - a_1^0(R_{0t}) = \sqrt{4S(R_{0t})/\tau_1},$$

where functions of the amplitude a_1^0 and S have been determined before [see Eqs. (22) and (29)]. Here in the domain $R_0>R_{0t}$ the equilibrium point is a focus and for $R_0>R_{0t}$ is a node. In a case of the PEV particle there exists only a lower bound which is determined by Eq. (28), and the state of equilibrium is a node.

A comprehensive investigation of the behavior of a plane rotator embedded in a fluid with a low electric conductivity and without shear flow was performed in [5]. The results of the present study for $\bar{\nu}=0$ recover the results obtained in [5]. Since without shear flow the only equilibrium orientation is $\theta=0$, changing the ratio of the semiaxes of the rotator which is characterized by a parameter β affects only the magnitude of the amplitude R_0 whereby different regimes of particle behavior are realized. The presence of a shear flow renders the parameter β more important since in this case it changes the location of the equilibrium states of a rotator and determines the existence of the stability domain and the distance between the upper and lower bounds of this domain. In contrast to [5] in this study we do not investigate rotator dynamics in the regions with weak and strong nonlinearity. It is expected that in the region with extremely low magnitude of parameter $\bar{\nu}$ in the vicinity of an upper boundary the behavior of the rotator is similar to the behavior determined in [5]. In the next section we investigate another equilibrium state which is related with a feasibility of rotation of a spheroid around its symmetry axis with a constant frequency.

IV. ROTATION OF SPHEROID AROUND AXIS OF SYMMETRY AND ITS STABILITY

Another stationary solution of Eqs. (13)–(19) describes rotation of spheroid around the symmetry axis with a constant angular velocity. In this section we analyze this solution following the same procedure as in the previous problem. First we determine this stationary solution and then investigate stability of particle rotation around the symmetry axis with a constant velocity in the case when particle inertia can be neglected. Consider a case when $\dot{\alpha}=$const. Equation (16) implies that in order to afford a constant rotation velocity, $\dot{\alpha}$, it is required that $\dot{\phi}=0$. The latter condition is satisfied only if $ctg\theta=0$. In this case $\Pi_3=0$. Stationary angle θ and Eq. (14) imply that $\phi=k\pi$. In this case the components of the unit vector directed along the symmetry axis are given by $\vec{e}_3=(0,\pm 1,0)$ (see Appendix A). Hence, rotation of a particle around the symmetry axis with a constant angular velocity can occur only when symmetry axis is directed along the flow vorticity, $\vec{\nabla}\times\vec{u}$. Hereafter, we assume that direction of vector $\vec{e}_3$ coincides with vector $\vec{\nabla}\times\vec{u}$, and hence $\theta=-\pi/2$, $\phi=0$. Equations (14)–(16) yield the following formulas:

$$\Pi_\theta=\Pi_\theta^k=-1/1+\omega^2,\quad \Pi_N^k=\omega/1+\omega^2, \tag{40}$$

and equation for $\omega=\dot{\alpha}\tau_1$ reads

$$\omega^3-\nu\omega^2+(1-R)\omega-\nu=0, \tag{41}$$

where $\nu=\bar{\nu}/2\tau_1$, and $R=\chi_1\tau_1R_0/M_{03}$. Parameters χ_1, τ_1, R_0, M_{03} are determined above [see Eqs. (5), (8), and (9)]. Equation (41) was investigated in [13] and also in [4]. In the following we will use the results obtained in these studies. In order to summarize the results of these studies let us introduce the parameters

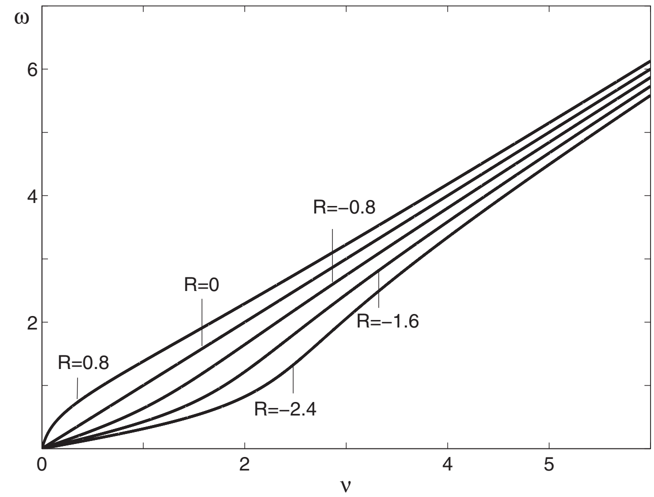

FIG. 7. Frequency of spheroid rotation vs rotation frequency of shear flow in the domain with one rotation regime.

$$a_1=9(1+R/2),\quad a_2=3(1-R),\quad d_1=-\nu(\nu^2+a_1),$$
$$d_2=|a_2-\nu^2|.$$

For the PEV particles, $\chi_1<0$ and, therefore, $R<0$. Positive values of parameter R correspond to NEV particles and negative values for PEV particles. In the case $R<0$ Eq. (41) has a unique solution:

$$\omega=\frac{\nu}{3}+y_1,\quad y_1=\frac{1}{3}\sum_{\alpha=\pm 1}\alpha(\sqrt{d_1^2+d_2^3}-\alpha d_1)^{1/3}. \tag{42}$$

In the range $\nu\ll 1$,

$$\omega=\nu/(1-R). \tag{43}$$

Expression (43) shows that velocity of rotation of the PEV particles ($R<0$) is smaller than rotation velocity of a fluid. Formula (42) is valid until condition $R<1$ is satisfied and $\nu^2<a_2$. When $R<1$ and $\nu^2<a_2$, similar to the previous case here also exists a single regime of rotation whereby rotation frequency is determined by the following formula:

$$\omega=\frac{\nu}{3}+y_2,\quad y_2=-\frac{1}{3}\mathrm{sgn}(d_1)\sum_{\alpha=\pm 1}(|d_1|+\alpha\sqrt{d_1^2-d_2^3})^{1/3}. \tag{44}$$

Formulas (42) and (44) describe rotation of not only the PEV particles but the NEV particles as well provided that the amplitude of the external electric field is small ($R<1$). In Fig. 7 we showed dependencies of the velocity of particle rotation ω vs frequency ν for various values of the parameter $R<1$. Negative and positive values of R characterize the behavior of the PEV and NEV particles, respectively. Inspection of this figure shows that velocity of PEV particles is always smaller than fluid velocity of rotation ($R=0$) while velocity of NEV particles is always larger than velocity of the fluid rotation. For $R>1$ the situation changes, and in this domain there exists the critical velocity of rotation of fluid, $\nu=\nu_{cr}(R)$:

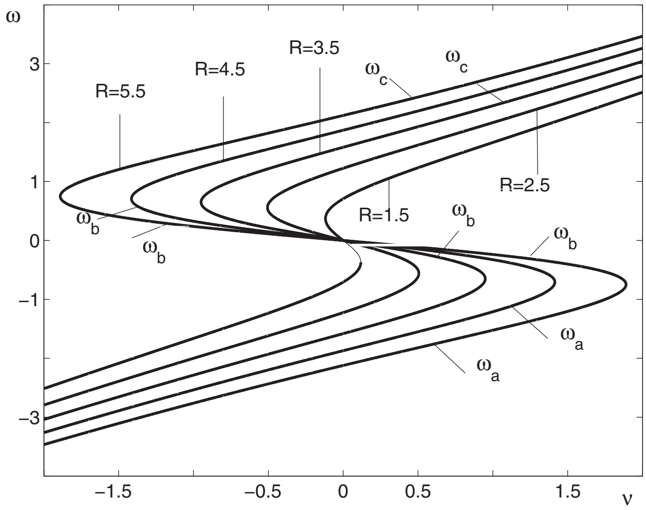

FIG. 8. Frequency of spheroid rotation vs rotation frequency of shear flow in the domain with several rotation regimes (a,b,c).

$$\nu_{cr}=\frac{\sqrt{R^2-20R-8+\sqrt{R(8+R)^3}}}{2\sqrt{2}}.$$

For $\nu>\nu_{cr}(R)$ velocity of particle rotation is determined by Eq. (44). When $\nu<\nu_{cr}(R)$ there are three roots, $y_a<y_b<y_c$:

$$y_a=-2\cos(\varphi_0),\quad y_b=2\cos\left(\varphi_0+\frac{\pi}{3}\right),$$

$$y_c=2\cos\left(\varphi_0-\frac{\pi}{3}\right),\quad \varphi_0=\frac{1}{3}\cos^{-1}\left(\frac{d_1}{\sqrt{d_2^3}}\right). \tag{45}$$

Possible velocities of rotation of a particle in this domain are given by the following formula:

$$\omega_i=\frac{\nu}{3}+\frac{\sqrt{d_2}}{3}y_i. \tag{46}$$

In Fig. 8 we showed the dependence of velocity of particle rotation for different regimes of rotation and different magnitudes of the amplitude of the external electric field, R. There exists a relation between velocities of particle rotation in various regimes, namely, $\omega_a(\nu)=-\omega_c(-\nu)$, $\omega_b(-\nu)=-\omega_b(\nu)$. The same relation holds for all other characteristics of particle behavior. Hence, if a particle is stable in a certain range of parameters in the regime a and is not stable in the same range of parameters in the regime c, then after inversing the direction of fluid rotation the particle will be stable in the regime c and unstable in the regime a. Since the regime b is always unstable [13] we do not consider it. The critical frequency, $\nu_{cr}(R)$, separated between a frequency domain $\nu<\nu_{cr}(R)$ with three regimes of rotation and a frequency domain $\nu>\nu_{cr}(R)$ with a single rotation regime. In the latter domain the velocity of rotation of the NEV particles is significantly higher than the velocity of fluid rotation in the same direction. Hereafter, we will call this regime a basic regime of rotation. When $\nu>0$ this regime corresponds to the regime c, and for $\nu<0$ it corresponds to the regime a.

Now let us consider stability of different particle rotation regimes around its axis when the particle orientation deviates from the initial orientation, $\vec{e}_3=(0,-1,0)$. The angles $\theta=-\pi/2+\theta'$ and $\phi=\phi'$ correspond to small deviations from this orientation. Linearization of Eqs. (14)–(16), (18), and (19) in the vicinity of the initial orientation yields a system of three equations with respects to the parameters θ',ϕ',Π_3'. The parameters α,Π_N,Π_θ do not include perturbations of the first order with respect to small deviations, θ' and ϕ'. Stability of the system for perturbations of the initial orientation of the rotator is determined by the condition Re $\gamma<0$. The increment γ satisfies the following equation:

$$\gamma^3+\bar{a}_1\gamma^2+\bar{a}_2\gamma+\bar{a}_3=0, \tag{47}$$

where

$$\bar{a}_1=\frac{1}{\tau_3}+\frac{R_0}{M_{01}}(F_1-\chi_1/Z),\quad \bar{a}_2=\bar{a}_2^0+\tau_3\bar{a}_3,$$

$$\bar{a}_3=\left[\frac{\nu}{2}(1-\Gamma_1)+\frac{R_0}{M_{01}Z}\chi_1\omega\right]\left[\frac{\nu}{2\tau_3}(1+\Gamma_1)\right], \tag{48}$$

$$\bar{a}_2^0=\frac{R_0(\chi_3+F_1-\chi_1/Z)}{M_{01}\tau_3}.$$

Here ω is one of the roots given by Eqs. (41) and (45) and $Z=1+\omega^2$. According to Routh-Hurwitz criterion, the condition of stability, $\text{Re}(\gamma)<0$, is satisfied if and only if

$$\bar{a}_1>0,\quad \bar{a}_1\bar{a}_2-\bar{a}_3>0,\quad \bar{a}_3>0. \tag{49}$$

Formulas (48) imply that $\bar{a}_3>0$ when $\omega>0$. Therefore, instability in the regime c can be associated only with violation of the first two conditions in Eqs. (49), while in the regime a the stability condition $\bar{a}_3>0$ is easily violated. The latter assertion is related with different behavior of the parameter Z in these two regions. In the regime a, $Z=Z_a(\nu,R)$ is located in the domain $Z_m(R)<Z_a(\nu,R)<R$, where

$$Z_m(R)=R(\sqrt{1+8/R}-1)/2. \tag{50}$$

In the regime c, $Z_c(\nu,R)>R$ and it has no upper bound. Let us demonstrate that the third condition in Eq. (49) is easily violated in the regime a for $\nu>0$, and, correspondingly, it is violated in the regime c for $\nu<0$. Consider the behavior of $Z_{a,c}(\nu,R)$ in the vicinity of a bifurcation threshold, $R=1$. Since near the bifurcation threshold $R-1\ll 1$, then $\nu_{cr}(R)=2(R-1)^{3/2}/3\sqrt{3}$ (see [4]). The domain of coexistence of different regimes and, consequently, of the regime a is determined by the condition, $\nu<\nu_{cr}(R)$. In the latter domain $\nu\ll(R-1)^{1/2}$ and, therefore, the formula for $Z_{a,c}(\nu,R)$ can be written as follows:

$$Z_{a,c}=R\mp\frac{\nu R}{\sqrt{R-1}}. \tag{51}$$

Formula (51) is derived by using Eq. (41) for formulating the equation for the function Z and solving it by iterations. The first nonvanishing approximation with respect to the parameter $\nu/(R-1)^{1/2}$ yields Eq. (51). Equation (41) implies that

$\omega/Z=\nu/(Z-R)$. Then the third condition in Eqs. (49) and (51) yields the following inequality:

$$\nu > \frac{2(R-1)^{1/2}M_{03}}{(1-\Gamma_1)M_{01}\tau_1}. \tag{52}$$

Inequality (52) cannot be satisfied in the domain of coexistence of the different regimes, $\nu<\nu_c(R)$, in the vicinity of a threshold $R=1$ except for the range of large β when $M_{03}/M_{01}\to 0$.

Note that Eq. (51) is satisfied not only in the above indicated domain but also in all cases when $\nu \ll R^{1/2}$. Therefore, shear flow causes destabilization of the regime a. Numerical analysis shows for $0<\nu<0.1\nu_c(R)$ and $R>1$ the frequencies of rotation $\omega_{a,c}$ is given by simple formulas, $\omega_{a,c} = \mp\sqrt{Z_{a,c}-1}$, where $Z_{a,c}(\nu,R)$ is determined by Eq. (51). Substitution $\nu\to-\nu$ in these formulas must be accompanied by the replacement of subscripts, $a\to c$.

Now let us investigate the first condition in Eq. (49) for the regime c assuming that $\nu>0$. For convenience of presentation consider first the case when $\nu=0$. It was demonstrated in [14] that for $\nu=0$ there is a range of the amplitude of the external electric field whereby two stationary regimes coexist. The first one is the regime of static orientation (RSO) which was considered in Sec. III and the regime of perpendicular orientation (RPO) whereby the particle rotates around its symmetry axis. Condition of stability of RSO was analyzed in the previous section. Condition for stability RPO for $\nu=0$ is implied by conditions (49) and Eq. (48). Conditions (49) can be rewritten here as $\bar{a}_1>0$ and $\bar{a}_2^0>0$, where the first inequality determines the upper bound of stability and the second the lower bound. Hence, we obtain

$$\frac{\chi_1}{\chi_3+F_1} < R < \left(\frac{\tau_1 M_{01}}{\tau_3 M_{03}} - 1\right)\frac{\chi_1}{|F_1|}. \tag{53}$$

Inequalities (53) are identical to Eqs. (28) and (29) in [14] (the difference is only in notations). Instability domain for RSO is determined by the condition, $R_0>R_{0c}(0)$ [see Eq. (31b)] or by the inequality $R>(M_{01}/M_{03})(\chi_1/F_1+\chi_3)$. Consequently, the domain of coexistence of ROS and RPO is given by the following formula:

$$\frac{\chi_1}{\chi_3+F_1} < R < \min\left\{\frac{M_{01}}{M_{03}}\frac{\chi_1}{F_1+\chi_3}, \left(\frac{\tau_1 M_{01}}{\tau_3 M_{03}} - 1\right)\frac{\chi_1}{|F_1|}\right\}. \tag{54}$$

Since for a prolate spheroid $M_{01}/M_{03}>1$, this domain always exists. Note also that for a prolate spheroid $\chi_1/(\chi_3+F_1)>1$. When the following inequality is satisfied:

$$\left(\frac{\tau_1}{\tau_3} - \frac{M_{03}}{M_{01}}\right) < \frac{1}{\dfrac{\chi_3}{|F_1|} - 1}, \tag{55}$$

the RPO is the first to lose its stability. Such situation occurs when, e.g., $\kappa_\varepsilon \gg 1$. When the inverse inequality is satisfied the RSO is the first to lose its stability. Such situation occurs when, e.g., $\kappa_\varepsilon \ll 1$ $(\varepsilon_1\to\varepsilon_2)$. This scenario for the behavior of the spheroid is valid only in the case $\nu=0$. The situation drastically changes when $\nu\neq 0$. Indeed, the presence of the arbitrary weak shear flow removes degeneracy, i.e., if at $\nu=0$ vector $\vec{e}_3$ could be directed in the arbitrary direction normal to the electric field, $\vec{e}_3=(l_3,m_3,0)$, when $\nu\neq 0$ due to the instability of the regime a there exists a unique direction $\vec{e}_3=(0,1,0)$. If there exists a domain where the regime a is stable the additional orientation, $\vec{e}_3=(0,-1,0)$, is possible. In order to avoid misunderstanding it must be emphasized that the effect of removal of the degeneracy is not related with the shape of a spheroid and remains for a spherical particle. The existence of the critical velocity of the shear flow which is determined by the parameter $\bar{\nu}_b$ (see Sec. III) imposes an additional constraint on the range of coexistence of two different regimes (RSO and RPO) simultaneously because the magnitude of the parameter $\bar{\nu}_b$ is small and for $\nu>\bar{\nu}_b\tau_1/2$ the RSO regime does not occur. The latter assertion by itself restricts the likelihood of the existence of these regimes for $\nu>0$. Analysis of the more general instability when the orientation of the rotation plane varies with time is quite involved since it requires investigating the algebraic equation of the sixth order, and this problem is not considered in this study. Hitherto we neglected particle inertia. Taking into account inertia of the particle leads to other mechanisms of instability which in the case without shear flow were investigated in [4,12,13]. The presence of a shear flow introduces new facets in the picture which are considered in the next section.

V. INERTIAL INSTABILITY OF ROTATION OF A SPHEROID AROUND THE AXIS OF SYMMETRY

In the analysis of inertial instability it is assumed that the axis of the spheroid is directed along the vorticity vector of a shear flow, $\vec{e}_3=(0,\pm 1,0)$. Taking into account inertia of particle rotating only around the axis of symmetry we arrive at the following system of equations:

$$\dot{\omega} = p(\nu-\omega+R\Pi_N), \quad \dot{\Pi}_N = -\Pi_N - \omega\Pi_\theta,$$

$$\dot{\Pi}_\theta = -\Pi_\theta + \omega\Pi_N - 1, \tag{56}$$

where $p=2\eta V M_{03}/I$ and I is a moment of inertia of spheroid relative to the symmetry axis [20].

Stationary solution of this system of equations is given by formulas (40), (42), (44), and (45). For investigating stability of this solution let us linearize this system in the vicinity of the stationary solution. The system of equations for the first-order corrections $(\omega^1,\Pi_N^1,\Pi_\theta^1)$ reads

$$\dot{\omega}^1 + p\omega^1 = pR\Pi_N^1, \quad \dot{\Pi}_N^1 + \Pi_N^1 = \frac{\omega^1}{Z} - \omega\Pi_\theta^1,$$

$$\dot{\Pi}_\theta^1 + \Pi_\theta^1 = \frac{\omega}{Z}\omega^1 + \omega\Pi_N^1, \tag{57}$$

where as before $Z=1+\omega^2$ and ω is of the roots given by Eqs. (42), (44), and (45). The spectrum of eigenvalues of Eq. (57) is determined by the following equation:

$$\gamma^3 + r_1\gamma^2 + r_2\gamma + r_3 = 0, \tag{58}$$

where $r_1=2+p$, $r_2=Z+2p-Rp/Z$, and $r_3=pZ-Rp(1-\omega^2)/Z$.

Condition of stability (Re $\gamma<0$) yields a system of inequalities similar to Eq. (49). Condition $r_1r_2-r_3>0$ implies that $\Delta_1(Z)=Z^2+2p(1+p/2-R/4)Z-Rp^2/2>0$, and condition $r_3>0$ implies $\Delta_2(Z)=Z^2+RZ-2R>0$. This system of inequalities we comprehensively investigated in [13]. In the latter study it was showed that stability of branches a and c is determined by the following condition:

$$Z_k(\nu,R) > Z_1(p,R), \quad k=a,c, \tag{59}$$

where

$$Z_1(p,R) = p[\sqrt{(1+p/2-R/4)^2+R/2}-(1+p/2-R/4)]. \tag{60}$$

Skipping the details which can be found in [13] it must be emphasized that all conclusions made in the latter study concerning particle behavior in the regime a are valid for the problem considered in this study for the regime c and all the conclusions in [13] on particle behavior in regime c are valid for the regime a in the present study. Therefore, along with the instability which was studied in Sec. IV, there exists an additional inertial instability that causes contraction of the domains of rotation. Since $Z_a(\nu,R)<R$, the condition $Z_1(p,R)>R$ determines the domain of the parameter p for given R whereby in the interval $p_-(R)<p<p_+(R)$, where

$$p_\pm(R) = 2[R/4-1 \pm \sqrt{(R/4-1)^2-R/2}], \tag{61}$$

the regime a is always unstable. When $Z_1(p,R)<R$, the regime a is stable if the rotation frequency of a shear flow ν satisfies the following condition:

$$\nu < \nu_{cr}^a = \sqrt{[Z_1(p,R)-1][R-Z_1(p,R)]/Z_1(p,R)}. \tag{62}$$

For $\nu>\nu_{cr}^a$ regime a is unstable. Therefore, instability of orientation in the regime a leads to the instability of this regime in the vicinity of a bifurcation threshold while the inertial instability occurs far away from the threshold and contributes to additional instability of the regime a. Since $Z_c(\nu,R)>R>Z_a(\nu,R)$, the regime c is always stable in the domain where the regime a is stable. Inertial instability of the regime c occurs only when $R>8+2\sqrt{12}$ since only under this condition $Z_1(p,R)>R$. In this case if $\nu>\nu_{cr}^c=\sqrt{[Z_1(p,R)-1][Z_1(p,R)-R]/Z_1(p,R)}$, then the regime c is always stable. A comprehensive study of this problem was conducted in [13].

VI. CONCLUSIONS

We investigated the effect of shear flow on the stationary regimes of rotation and static orientation of a spheroidal particle. In the case of static orientation, shear flow alters the angle of orientation which strongly depends upon the internal and external parameters of the system. Stability of static orientation (RSO) depends on the internal parameters of the system, i.e., whether the particle is a PEV particle when condition (1) is satisfied or a NEV particle whereby this condition is violated. In the latter case the range of the amplitude of the external electric field where static orientation regime is stable contracts because of the emergence of the upper bound for the field amplitude which strongly depends upon the rotation velocity of a shear flow. Shear flow also strongly affects the feasibility of coexistence of the two stationary regimes, RSO and RPO. The main effect of the shear flow is removal of the degeneracy. The latter results in that rotation of particle around the symmetry axis can occur only when this axis is directed parallel to the vorticity vector while without shear flow the axis of symmetry of particle can be directed along the arbitrary direction which is perpendicular to the external electric field. The second effect of a shear flow is the emergence of the critical frequency $\bar{\nu}_b$ whereby at frequencies $\bar{\nu}>\bar{\nu}_b$ static orientation of particle in the plane spanned by vectors $\vec{E}$ and $\vec{u}$ is not possible. Shear flow leads to the expansion of the domain of external parameters where rotation around the symmetry axis oriented along the vorticity of a shear flow is stable and causes contraction of this domain when the direction of the angular velocity of a particle is opposite to the vorticity vector.

APPENDIX A

According to Eq. (36) in Ref. [13] the components of fluid flow induced torque $\vec{M}=(L,M,N)$ can be written as follows (for details see [13], Eq. (36)):

$$L = \frac{16\pi\mu(b^2+c^2)}{3(b^2\beta_0+c^2\gamma_0)}\left[\left(\frac{b^2-c^2}{b^2+c^2}\right)f+\xi-\omega_1\right],$$

$$M = \frac{16\pi\mu(a^2+c^2)}{3(c^2\gamma_0+a^2\alpha_0)}\left[\left(\frac{c^2-a^2}{a^2+c^2}\right)g+\eta-\omega_2\right],$$

$$N = \frac{16\pi\mu(a^2+b^2)}{3(a^2\alpha_0+b^2\beta_0)}\left[\left(\frac{a^2-b^2}{a^2+b^2}\right)h+\zeta-\omega_3\right], \tag{A1}$$

where a,b,c are the lengths of the semiaxes of the ellipsoid in the directions of the vectors $\vec{e}_1,\vec{e}_2,\vec{e}_3$. In the present study a,b,c are denoted by a_1,a_2,a_3, respectively. Parameters $\alpha_0,\beta_0,\gamma_0$ in Eq. (A1) are determined by Eq. (9) in [13]. Parameters f,g,h in Eq. (A1) for the flow velocity (see Eq. (40) in [13])

$$\vec{u} = \vec{k}\kappa y \tag{A2}$$

can be written using the following notations: $f=c_1$, $g=c_2$, $h=c_3$, as $c_i=\frac{\kappa}{2}e_{ikl}^2 m_k n_l$. Parameters ξ,η,ζ in Eq. (A1) for flow velocity given by Eq. (A2) are the components of the vector $\vec{\nabla}\times\vec{u}/2=\kappa\vec{i}/2$ along the directions $\vec{e}_1,\vec{e}_2,\vec{e}_3$. Taking into account the difference in the formulas for velocity, Eqs. (1) and (A2), using the new notation for the dynamic viscosity of a host fluid ($\mu\to\eta$) expressing parameters $\alpha_0,\beta_0,\gamma_0$ through the coefficient of depolarization d and considering the case of a spheroid ($a=b$) we arrive at the equations for torque in Sec. II.

In the analysis of dynamics of particle rotation it is convenient to express vectors $\vec{e}_i=(l_i,m_i,n_i)$ through Euler angles. To this end e_i can be written as $\vec{e}_i=(O_{i1},O_{i2},O_{i3})$, where O_{ik} are matrix elements of the operator $\hat{O}$:

$$\hat{O} = \hat{O}_3(\alpha)\hat{O}_2(\theta)\hat{O}_1(\phi), \tag{A3}$$

and

$$\hat{O}_1(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\hat{O}_2(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

$$\hat{O}_3(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A4}$$

Hence, we obtain the following formulas:

$$l_1 = \cos\alpha\cos\phi - \sin\alpha\sin\phi\cos\theta,$$

$$l_2 = -\sin\alpha\cos\phi - \cos\alpha\sin\phi\cos\theta,$$

$$l_3 = \sin\phi\sin\theta,$$

$$m_1 = \cos\alpha\sin\phi + \sin\alpha\cos\theta\cos\phi,$$

$$m_2 = -\sin\alpha\sin\phi + \cos\alpha\cos\theta\cos\phi,$$

$$m_3 = -\cos\phi\sin\theta,$$

$$n_1 = \sin\alpha\sin\theta, \quad n_2 = \cos\alpha\sin\theta, \quad n_3 = \cos\theta. \tag{A5}$$

APPENDIX B

The results derived in this study are based on the system of equations which includes Poisson equation (B1), equation of conservation of electric charge (B2), and equations which relate the principal electrodynamic variables in the problem [Eq. (B3)]:

$$\vec{\nabla}\cdot\vec{D} = \rho_{ex}, \tag{B1}$$

$$\frac{\partial\rho_{ex}}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0, \tag{B2}$$

where

$$\vec{D} = \varepsilon_0\varepsilon\vec{E}, \quad \vec{j} = \vec{j}_\sigma + \vec{j}_c, \quad \vec{j}_\sigma = \sigma\vec{E}, \quad \vec{E} = -\vec{\nabla}\varphi. \tag{B3}$$

Here ρ_{ex} is a density of the external electric charge which is induced by the external electric field at a particle-fluid interface due to finite electric conductivity of a system, $\vec{j}_c$ is a convective electric current caused by a macroscopic motion of the charged particle, $\vec{j}_c=\rho_{ex}\vec{v}_s$. The density of a free surface charge γ is determined by the following relations:

$$\int \rho_{ex} dV = \int \gamma dS, \quad \text{or} \quad \rho_{ex} = \gamma\delta(u)|\vec{\nabla}u|,$$

where $\delta(u)$ is Dirac's delta function; $u=F(x,y,z,t)$ is equation of a surface related with a velocity at the particle surface v_S by equation $\partial u/\partial t + v_S\nabla u=0$. In a case of a solid particle a whole-particle motion and, consequently, a convective current are eliminated by transfer to the coordinate system attached with a particle $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$. In this coordinate system Eqs. (B1)–(B3) are supplemented with boundary conditions $[\vec{N}\cdot\vec{D}]=\gamma$, $[\vec{N}\cdot\vec{j}_\sigma]=-\partial\gamma/\partial t$, where $[A]=A_+-A_-$, A_+ and A_- are values of A at the external and internal surfaces, respectively, and $\vec{N}$ is external unit normal vector. Since each of Eqs. (B1) and (B2) together with the corresponding boundary conditions determine the same potential φ, equating the potential found using Eq. (B1) to the potential found from Eq. (B2) yields the equation for electric charge relaxation [9,10]. The dipole moment $P(t)$ is determined by the following formulas:

$$\vec{P} = \int \gamma_T\vec{r}dS, \quad \gamma_T = \varepsilon_0[\vec{E}]\cdot\vec{N}, \tag{B4}$$

where integration is performed over the surface of spheroid. Equation (B4) is a straightforward extension of the definition of the dipole moment [16,17] for the case of a system with a finite electric conductivity, and for $\kappa_\sigma=0$ the dipole moment determined by Eq. (B4) coincides with the dipole moment calculated in [16,17] (compare Eq. (6) in the present study and formulas in [17], problem 200). Some studies of particle rotation in systems with small electric conductivity employ a different definition of the dipole moment. In [3] the dipole moment is defined as $\vec{P}=\int\gamma_{ex}\vec{r}dS$, $\gamma_{ex}=\varepsilon_0[\vec{D}]\cdot\vec{N}$. The latter definition of the dipole moment does not account for the polarization-induced dipole moment. This definition of the dipole moment does not yield erroneous results only in calculating the electric torque, $\vec{T}=\vec{P}\times\vec{E}$, acting at the sphere which was considered in [3] because the component of the torque $\vec{T}$ caused by the "instantaneous" polarization vanishes in the case of a sphere. Another definition of the dipole moment is used in [18] where $\vec{P}$ is defined by $\vec{D}=\varepsilon_\infty\vec{E}+\vec{P}$, where ε_∞ depends upon the geometry of the problem [see [18], Eqs. (1), (6), and (7)]. The difference between the latter definition of the dipole moment and Eq. (B4) employed in this study is the cause of difference in some formulas as indicated in the present study. Equations for electric charge relaxation (for details see [9,10]) yield the expression for the dipole moment, P_σ. Using Eq. (7) for the dipole moment P_σ we arrive at Eq. (8) in this study. Equations (7) and (8) are equivalent to the equations for the dipole moment which have been repeatedly used in the literature (see, e.g., [5,14,18]).

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